

Classical Euclidean wormhole solutions in Palatini $f(\tilde{R})$ cosmology

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Abstract

We study the classical Euclidean wormholes in the context of Palatini $f(\tilde{R})$ cosmology. We use the dynamical equivalence between $f(\tilde{R})$ gravity and scalar-tensor theories to construct a point-like Lagrangian in the flat FRW space time. We first show the dynamical equivalence between Palatini $f(\tilde{R})$ gravity and the Brans-Dicke theory with self-interacting potential, and then show the dynamical equivalence between the Brans-Dicke theory with self-interacting potential and the minimally coupled O'Hanlon theory. We show the existence of new Euclidean wormhole solutions for this O'Hanlon theory and, for an special case, find out the corresponding form of $f(\tilde{R})$ having wormhole solution. For small values of the Ricci scalar, this $f(\tilde{R})$ is in agreement with the wormhole solution obtained for higher order gravity theory $\tilde{R} + \epsilon \tilde{R}^2$, $\epsilon < 0$.

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1 Introduction

Classical wormholes are usually considered as Euclidean metrics that consist of two asymptotically flat regions connected by a narrow throat (handle). Wormholes have been studied mainly as instantons, namely solutions of the classical Euclidean field equations. In general, such wormholes can represent quantum tunneling between different topologies. They are possibly useful in understanding black hole evaporation [1]; in allowing nonlocal connections that could determine fundamental constants; and in vanishing the cosmological constant Λ [2]-[4]. They are even considered as an alternative to the Higgs mechanism [5]. Consequently, such solutions are worth finding.

The reason why classical wormholes may exist is related to the implication of a theorem of Cheeger and Glomol [6] which states that a *necessary* (but not sufficient) condition for a classical wormhole to exist is that the eigenvalues of the Ricci tensor be negative *somewhere* on the manifold. For example, the energy-momentum tensors of an axion field and of a conformal scalar field are such that, when coupled to gravity, the Ricci tensor has negative eigenvalues. However, for a (real) minimally coupled scalar field it is easy to show that the Ricci tensor can never be negative. It is shown that the (pure imaginary) minimally coupled scalar field, conformally coupled scalar field, a third rank antisymmetric tensor field and some special kinds of matter sources have wormhole solutions [7]-[11].

On the other hand, Euclidean wormhole solutions are provided for some higher-order gravity theories by using the conformal equivalence theorem [12]-[17]. This theorem shows that the space of solutions of the higher-order gravity theory is conformally equivalent to that of general relativity with a minimally coupled scalar field with self-interacting potential [18], [19].

The work in the present paper is similar in spirit to the works done by the authors in Refs.[12]-[17]: We use the dynamical equivalence between Palatini $f(\tilde{R})$ gravity and scalar-tensor theories to construct a point-like Lagrangian in the flat Friedmann-Robertson-Walker (FRW) universe [20, 21]. In so doing, we first show the dynamical equivalence between Palatini $f(\tilde{R})$ gravity and the Brans-Dicke theory with self-interacting potential. Then, we show the dynamical equivalence between the Brans-Dicke theory with self-interacting potential and the minimally coupled O'Hanlon theory in which the dynamics is completely endowed by the self interacting potential. We aim to show the existence of new Euclidean wormhole solutions for this O'Hanlon theory and then find out the possible corresponding forms of $f(\tilde{R})$ in Palatini formalism.

2 Dynamical equivalence between Palatini $f(R)$ gravity and minimally coupled O'Hanlon theory

The action of the Palatini $f(\tilde{R})$ theories takes the following form

$$S = \frac{1}{2k} \int d^4x \sqrt{-g} f(\tilde{R}), \quad (1)$$

where $f(\tilde{R})$ is a function of $\tilde{R} = g^{\mu\nu} R_{\mu\nu}(\tilde{\Gamma})$ and $\tilde{\Gamma}_{\mu\nu}^{\lambda}$ is the connection. This action depends on the two dynamical variables, namely metric and connection. Variation of Eq.(1) with respect

to the metric leads to

$$f'(\tilde{R})\tilde{R} - \frac{1}{2}f(\tilde{R})g_{\mu\nu} = 0, \quad (2)$$

where $f'(\tilde{R}) = df/d\tilde{R}$. The trace of Eq.(2) is

$$f'(\tilde{R})\tilde{R} - 2f(\tilde{R}) = 0, \quad (3)$$

and the variation of Eq.(1) with respect to the connection gives

$$(\sqrt{-g}f'(\tilde{R})g^{\mu\nu})_{;\lambda} = 0, \quad (4)$$

where $;$ denotes covariant derivative. Therefore, the connection is compatible with the new metric $h_{\mu\nu} = f'(\tilde{R})g_{\mu\nu}$ and we obtain

$$\tilde{R} = R + \frac{3}{2f'(\tilde{R})}\partial_\lambda f'(\tilde{R})\partial^\lambda f'(\tilde{R}) - \frac{3}{f'(\tilde{R})}\square f'(\tilde{R}), \quad (5)$$

where R is Ricci scalar constructed from the Levi-Civita connection of the metric $g_{\mu\nu}$. One can easily verify that the action (1) is dynamically equivalent to [20]

$$S = \frac{1}{2k} \int d^4x \sqrt{-g} (\Phi R + \frac{3}{2\Phi} \Phi_{;\mu} \Phi^{;\mu} - V(\Phi)), \quad (6)$$

where $\Phi = f'(\tilde{R})$, $V(\Phi) = \chi(\Phi)\Phi - f(\chi(\Phi))$ and $\tilde{R} = \chi(\Phi)$. This is the well-known action of Brans-Dicke theory with the coupling parameter equal to $-\frac{3}{2}$. We now use some suitable redefinitions followed by conformal transformations to cast the above Brans-Dicke theory into the minimally coupled scalar field theory.

First, we use the redefinition $\Phi \equiv \varphi^2$ by which the action (6) takes the following form

$$S = \frac{1}{2k} \int d^4x \sqrt{-g} (\varphi^2 R + 6\varphi_{;\mu} \varphi^{;\mu} - V(\varphi)). \quad (7)$$

This action is dynamically equivalent to

$$S = \frac{1}{2k} \int d^4x \sqrt{-g} (F(\varphi) R + \frac{1}{2} \varphi_{;\mu} \varphi^{;\mu} - U(\varphi)), \quad (8)$$

where $F(\varphi) = \frac{1}{12}\varphi^2$ and $U(\varphi) = \frac{1}{12}V(\varphi)$. Let us now consider a conformal transformation on the metric $g_{\mu\nu}$ [22]

$$\bar{g}_{\mu\nu} = e^{2\sigma} g_{\mu\nu}, \quad (9)$$

where $e^{2\sigma}$ is the conformal factor. The Lagrangian density in (8) then becomes

$$\begin{aligned} & \sqrt{-g} (F R + \frac{1}{2} \varphi_{;\mu} \varphi^{;\mu} - U) \\ &= \sqrt{-\bar{g}} e^{-2\sigma} (F \bar{R} - 6F \square_{\bar{\Gamma}} \sigma - 6F \sigma_{;\alpha} \sigma^{;\alpha} + \frac{1}{2} \bar{g}^{\mu\nu} \varphi_{;\mu} \varphi_{;\nu} - e^{-2\sigma} U), \end{aligned} \quad (10)$$

where $\bar{R}, \bar{\Gamma}$ and $\square_{\bar{\Gamma}}$ are the corresponding quantities with respect to the metric $\bar{g}_{\mu\nu}$ and connection $\bar{\Gamma}$, respectively. If we require the new theory in terms of $\bar{g}_{\mu\nu}$ to appear as a standard Einstein theory the conformal factor has to be related to F as

$$e^{2\omega} = 2F. \quad (11)$$

Using this relation, the Lagrangian density (10) becomes

$$\sqrt{-g}(FR + \frac{1}{2}\varphi_{;\mu}\varphi^{;\mu} - U) = \sqrt{-\bar{g}}\left(\frac{1}{2}\bar{R} + 3\square_{\bar{\Gamma}}\sigma + \frac{3F_{\varphi}^2 - F}{4F^2}\varphi_{;\alpha}\varphi^{;\alpha} - \frac{U}{4F^2}\right). \quad (12)$$

By introducing a new scalar field $\bar{\varphi}$ and the potential \bar{U} , respectively, defined by

$$\bar{\varphi}_{;\alpha} = \sqrt{\frac{3F_{\varphi}^2 - F}{4F^2}}\varphi_{;\alpha}, \quad \bar{U}(\bar{\varphi}(\varphi)) = \frac{U}{4F^2}, \quad (13)$$

we obtain [22]

$$\sqrt{-g}(FR + \frac{1}{2}g^{\mu\nu}\varphi_{;\mu}\varphi^{;\mu} - U) = \sqrt{-\bar{g}}\left(\frac{1}{2}\bar{R} + \frac{1}{2}\bar{\varphi}_{;\alpha}\bar{\varphi}^{;\alpha} - \bar{U}\right), \quad (14)$$

where the r.h.s. is the usual Einstein-Hilbert Lagrangian density subject to the metric $\bar{g}_{\mu\nu}$, plus the standard Lagrangian density of the scalar field $\bar{\varphi}$. This form of the Lagrangian density is usually referred to the *Einstein frame*. Therefore, we realize that any nonstandard coupled theory of gravity with scalar field, in the absence of ordinary matter, is conformally equivalent to the standard Einstein gravity minimally coupled with scalar field provided we use the conformal transformation (9) together with the definitions (11) and (13). However, if we put $F(\varphi) = \frac{1}{12}\varphi^2$ in the first definition in Eq.(13) we obtain $\bar{\varphi}_{;\alpha} = 0$ which leads to the following action

$$\mathcal{S} = \int d^4x \sqrt{-\bar{g}} \left(\frac{1}{2}\bar{R} - \bar{U} \right). \quad (15)$$

This is known as the O'Hanlon action where the dynamics is completely endowed by the self interacting potential.

3 Classical Euclidean wormholes in O'Hanlon theory

In this section, we look for the wormhole solutions in the system (15) of minimally coupled scalar field with the lagrangian density

$$\mathcal{L} = \frac{1}{2}\bar{R} - \bar{U}. \quad (16)$$

where $U(\bar{\varphi})$ is a self-interacting potential. We do not specify *a priori* the form of the potential, and by analyzing the field equations and the corresponding wormhole solutions we may use the conformal equivalence discussed in the previous section to go in the opposite direction and

obtain the corresponding wormhole solutions in Palatini $f(R)$ gravity. The Einstein equations of motion are obtained as

$$\bar{R}_{\mu\nu} = \bar{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{T}, \quad (17)$$

where the energy-momentum tensor and its trace are given respectively by

$$\bar{T}_{\mu\nu} = \bar{g}_{\mu\nu}\bar{U}(\bar{\varphi}), \quad (18)$$

$$\bar{T} = 4\bar{U}(\bar{\varphi}). \quad (19)$$

Putting (18) and (19) in Eq.(17) we obtain

$$\bar{R}_{\mu\nu} = -\bar{g}_{\mu\nu}\bar{U}(\bar{\varphi}). \quad (20)$$

It is seen that for the positive definite Euclidean metric $\bar{g}_{\mu\nu}$ the Ricci tensor $\bar{R}_{\mu\nu}$ has negative eigenvalues if and only if the potential $\bar{U}(\bar{\varphi})$ is positive, and consequently wormhole solutions may exist in this system if and only if the following two conditions hold

$$\bar{U}(\bar{\varphi}) > 0, \quad (21)$$

$$-\bar{g}_{\mu\nu}\bar{U}(\bar{\varphi}) < 0. \quad (22)$$

For the flat Friedmann-Robertson-Walker universe the Euclidean metric $\bar{g}_{\mu\nu}$ is written as

$$dS^2 = dt^2 + a^2(t)d^2\Omega_3, \quad (23)$$

where $d^2\Omega_3$ is the line element on the three-sphere. The field equation for the variable a is obtained as

$$\dot{a}^2 = 1 - a^2\bar{U}(\bar{\varphi}), \quad (24)$$

where an overdot denotes d/dt . Now, we look for the wormhole solutions for the equation (24). It is generally believed that a wormhole has two asymptotically flat regions connected by a throat at which $\dot{a} = 0$ and it is described by an expression of the form

$$\dot{a}^2 = 1 - \frac{C}{a^n}, \quad (25)$$

where C is a constant. In order to have an asymptotic Euclidean wormhole it is necessary that \dot{a}^2 remains positive at large a , and this requires $n > 0$. Comparison of the equations (24) and (25) shows that it is possible to choose a suitable form of the potential $\bar{U}(\bar{\varphi})$ so that Eq.(24) represents a wormhole. Therefore, the existence of wormholes for O'Hanlon theory is established. Now, we wish to look for the corresponding wormholes in the Palatini $f(\tilde{R})$ theory. In so doing, we may rewrite the potential in the following form

$$\bar{U}(\bar{\varphi}) = \frac{U}{4F^2} = 3\frac{V(\varphi)}{\varphi^4} = 3\frac{\tilde{R}\Phi - f(\tilde{R})}{\Phi^2} = 3\frac{\tilde{R}f' - f(\tilde{R})}{f'^2}. \quad (26)$$

In order this potential, after inclusion in Eq.(24), could represent a wormhole we should take the following equation

$$\frac{\tilde{R}f' - f(\tilde{R})}{f'^2} = \frac{C}{a^{n+2}}. \quad (27)$$

One may wish to rewrite this equation as a first order differential equation for f as a function of a or \tilde{R} . In both cases we need to express the Ricci scalar \tilde{R} in terms of the scale factor a . Using the metric (23), the scalar curvature is obtained as

$$\tilde{R} = 6 \left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right]. \quad (28)$$

If we are to consider (25) as the wormhole solution, then the Ricci scalar should take the following form

$$\tilde{R} = 6 \left[\frac{C}{a^{n+2}} \left(\frac{n}{2} - 1 \right) + \frac{1}{a^2} \right], \quad (29)$$

where the values of n and C may fix the sign of the Ricci scalar.

It is of valuable practice to obtain the wormhole solutions in $f(\tilde{R})$ theory whose defining equations are the same as the well-known wormhole solutions in conventional Einstein-Hilbert theory. For the sake of simplicity we first take $n = 2$. This case corresponds to the typical known wormhole of conformal scalar field coupled with Einstein-Hilbert action [23, 24, 25]. The Ricci scalar and f' then become respectively

$$\tilde{R} = \frac{6}{a^2}, \quad (30)$$

$$f' = \frac{df}{d\tilde{R}} = \frac{df}{da} \frac{da}{d\tilde{R}} = -\frac{a^3}{12} \frac{df}{da}. \quad (31)$$

Putting Eqs.(30) and (31) in (27) we obtain the following first order differential equation

$$a^6 \left(\frac{df}{da} \right)^2 + Aa^5 \left(\frac{df}{da} \right) + 2Aa^4 f = 0. \quad (32)$$

We may also use (32) and its derivative d/da to obtain the second order differential equation

$$Aa^5 \frac{d^2 f}{da^2} + 2a^6 \frac{df}{da} \frac{d^2 f}{da^2} + 2a^5 \left(\frac{df}{da} \right)^2 + 3Aa^4 \frac{df}{da} = 0, \quad (33)$$

where $A = \frac{72}{C}$. Alternatively, we may use Eq.(30) in Eq.(27) to express a in terms of \tilde{R} . In doing so, we obtain the following first order differential equation

$$\tilde{R}^2 \left(\frac{df}{d\tilde{R}} \right)^2 - \tilde{R} \left(\frac{df}{d\tilde{R}} \right) + f = 0. \quad (34)$$

Equivalently, using the derivative $d/d\tilde{R}$ of Eq.(34) we obtain the following second order differential equation for f in terms of \tilde{R}

$$\frac{d^2 f}{d\tilde{R}^2} \left(1 - 2\tilde{R} \frac{df}{d\tilde{R}} \right) - 2 \left(\frac{df}{d\tilde{R}} \right)^2 = 0. \quad (35)$$

Since we are interested in the explicit function $f(\tilde{R})$, we consider Eq.(35) whose solution may be obtained either in the parametric form

$$\begin{cases} f(T) = \frac{1}{4} - \frac{1}{4} \ln(T)^2 - \frac{C}{2} \ln(T) - \frac{1}{4} C^2, \\ R(T) = \frac{1}{2T} [1 - \text{sgn}[\ln(T) + C] \ln(T) - \text{sgn}[\ln(T) + C] C], \end{cases} \quad (36)$$

$$\begin{cases} f(T) = \frac{1}{4} - \frac{1}{4} \ln(T)^2 - \frac{C}{2} \ln(T) - \frac{1}{4} C^2, \\ R(T) = \frac{1}{2T} [1 + \text{sgn}[\ln(T) + C] \ln(T) + \text{sgn}[\ln(T) + C] C], \end{cases} \quad (37)$$

with C being a constant and “ sgn ” denoting the *sign function*, or in the explicit form¹

$$f(\tilde{R}) = -\frac{1}{4} \left[\text{LambertW} \left(-2C_1 \tilde{R} \right) \right]^2 - \frac{1}{2} \text{LambertW} \left(-2C_1 \tilde{R} \right) + C_2, \quad (38)$$

where C_1 and C_2 are constants and LambertW is the LambertW function. For small values of the Ricci scalar \tilde{R} we obtain, modula the constant C_1 , the following form

$$f(\tilde{R}) \simeq \tilde{R} - C_1 \tilde{R}^2 + \frac{C_2}{C_1}. \quad (39)$$

It is worth noting that the above solution, modula the constant term C_2/C_1 , is the wormhole found by *Fukutaka et al* in the higher order gravity theory $\tilde{R} + \epsilon \tilde{R}^2$, $\epsilon < 0$, for closed Friedman-Robertson-Walker universe [26]. This interesting agreement may confirm the correctness of the general form $f(\tilde{R})$ given by Eq.(38) for which we expect classical Euclidean wormholes.

Next, we may take $n = 4$ which corresponds to the *axion* field as the matter source coupled with Einstein-Hilbert action that leads to the Giddings-Strominger wormhole [7]. The Ricci scalar and f' then become respectively

$$\tilde{R} = 6 \left[\frac{C}{a^6} + \frac{1}{a^2} \right], \quad (40)$$

$$f' = \frac{df}{d\tilde{R}} = \frac{df}{da} \frac{da}{d\tilde{R}} = -\frac{1}{6} \frac{df}{da} \left[\frac{a^7}{6C + 2a^4} \right]. \quad (41)$$

Putting Eqs.(40) and (41) in (27) we obtain the following first order differential equation

$$Aa^8 \left(\frac{df}{da} \right)^2 + a(C + a^4)(6C + 2a^4) \frac{df}{da} + (6C + 2a^4)^2 f = 0. \quad (42)$$

¹The LambertW function satisfies

$$\text{LambertW}(x) \exp(\text{LambertW}(x)) = x,$$

and it has infinite number of branches for each (nonzero) value of x while exactly one of these branches is analytic at zero. For small values of x we have $\text{LambertW}(x) \simeq x$.

We may also use (42) and its derivative d/da to obtain the second order differential equation

$$a(C + a^4)(6C + 2a^4)\frac{d^2 f}{da^2} + 2Aa^8\frac{df}{da}\frac{d^2 f}{da^2} + 8Aa^7\left(\frac{df}{da}\right)^2 + (42C^2 + 64Ca^4 + 22a^8)\frac{df}{da} + a^3(96C + 32a^4)f = 0. \quad (43)$$

where $A = \frac{36}{C}$. Unfortunately, neither the first order nor the second order differential equations gives exact solutions (at least using the available mathematical software like Maple).

Alternatively, if we wish to use Eq.(40) to express a in terms of \tilde{R} and construct a differential equation like Eqs.(34), (35), then we obtain more complicate differential equations with no exact solution, so we ignore to follow seriously this case.

Conclusions

Any wormhole solution is of particular importance from macroscopic and microscopic points of view. In particular, very small wormholes are studied as instantons, namely the saddle points in the Euclidean path integrals. So, one can use them to give a semi-classical treatment in the dilute wormhole approximation where the interaction between the large scale ends of wormholes is neglected. On the other hand, since the black holes evaporate in theories with any reasonable matter content, then any new wormhole solution may provide a new contribution for black hole evaporation. In the same way, any new wormhole solution is supposed to play its own important role in vanishing the cosmological constant. Taking into account the importance of new wormhole solutions and motivated by the existence of Euclidean wormhole solutions for some higher-order gravity theories we have studied the classical Euclidean wormhole solutions for modified general $f(\tilde{R})$ theories of gravity in Palatini formalism. We tried to establish a dynamical equivalence between $f(\tilde{R})$ gravity and scalar-tensor theories to construct a point-like Lagrangian in the flat FRW space time. In this regard, we first used the dynamical equivalence between Palatini $f(\tilde{R})$ gravity and the Brans-Dicke theory with self-interacting potential, and then showed the dynamical equivalence between the Brans-Dicke theory with self-interacting potential and the minimally coupled O'Hanlon theory. We realized the existence of new Euclidean wormhole solutions for this O'Hanlon theory and for an special case we obtained the corresponding (wormhole) form of $f(\tilde{R})$ which, for small \tilde{R} , is in agreement with the wormhole solution obtained for higher order gravity theory $\tilde{R} + \epsilon\tilde{R}^2$, $\epsilon < 0$.

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